The motion of buoyant elements in turbulent surroundings

By J. S. TURNER

C.S.I.R.O. Radiophysics Laboratory, Sydney, Australia†

(Received 28 August 1962 and in revised form 26 November 1962)

In this paper a theoretical model of the motion of isolated buoyant elements in turbulent surroundings is introduced, which takes into account both the growth due to turbulent entrainment and a loss of buoyant fluid to the environment. On dimensional grounds the outflow velocity is taken to be constant and proportional to some characteristic turbulent velocity in the environment, while the entrainment velocity is proportional to the upward velocity of the element. Numerical solutions of the resulting non-dimensional equations of motion are presented, corresponding to a wide range of stabilities. Typically, an element in stable, neutral or moderately unstable surroundings at first grows and then is eroded away, but at a certain value of a stability parameter γ elements become absolutely unstable and continue to grow and rise indefinitely. The value of γ is extremely sensitive to the level of turbulence in the environment, which could therefore exert a controlling influence on the growth of buoyant elements in unstable conditions; large elements are more likely to grow when the level of turbulence is low.

Laboratory experiments have been carried out in order to test one of the predictions of this theory, the form of the dependence of the height attained on total buoyancy and level of turbulence in uniform surroundings. The agreement is good, and numerical comparison of theory and experiment suggests that the assumed outflow velocity is of the same order as, but somewhat less than, the r.m.s. turbulent velocity.

1. Introduction

In previous discussions of the motion of buoyant elements or parcels rising through the atmosphere, two rather different theoretical models have been used. In both of them the mixing with the surroundings has been found to have an important effect on the motion, though this mixing is supposed to occur in different ways in the two cases.

The first type of theory has considered a turbulent element or 'thermal' moving through still surroundings (see, for example, Morton, Taylor & Turner 1956; Scorer 1957). In this case the mixing is due entirely to the motions within the element which are produced by the action of buoyancy forces, and the growth

† At present on leave, working as Rossby Memorial Fellow at the Woods Hole Oceanographic Institution, Woods Hole, Massachusetts U.S.A.

of the turbulent region by incorporation of external fluid is taken into account. Dimensional arguments lead to the result that the radius of thermals will increase linearly with distance, which also implies that the entrainment velocity is everywhere proportional to the upward velocity. This has been verified by laboratory experiment over a wide range of stability conditions in the environment.

The second method of approach, due originally to Priestley (1953), regards the buoyant element as a region of constant size, or 'open parcel', which is interchanging fluid with its surroundings at a rate governed by the level of turbulence in the environment. In the shear layers near the ground, turbulence produced nearly independently of the motion of buoyant elements will certainly have an important influence on the motion, and the neglect of this effect in the thermal model is certainly a matter 'of mathematical convenience rather than physical belief' (Priestley 1960). Open parcels can lose momentum and buoyancy to their surroundings, at a rate which depends on their size, and elements can be shown to behave differently in the same environment merely because of a difference in size.

Both these types of theory therefore have features which one feels should be included in a proper description of buoyant motion in turbulent surroundings, but both omit other important facts. The idea that mixing into a turbulent element takes place at a rate proportional to the scale of mean velocity seems well supported in still surroundings; it is equally certain that there will be some removal of buoyant fluid in turbulent surroundings. The open parcel model allows for the outward flux in a realistic way for elements of fixed size in which the turbulence can be regarded as part of the environmental turbulence, but it therefore at the same time makes the inflow no longer dependent on the velocity of the element. Another disadvantage is that, although the mixing rate depends on size, there is nothing in the theory which indicates how the appropriate size should be chosen, or how it may vary with time.

Each of these theories could of course still be relevant at a different stage of the history of a single element. When its velocity is high, the environmental turbulence will be relatively unimportant and the thermal model would be appropriate, and when the turbulent velocities inside and outside are comparable, something like an open parcel will be necessary. One approach to the problem of describing the whole motion would be to make a superposition of the two separate theories, and this is in fact the method used by Priestley (1956) in his theory of bent-over buoyant plumes.

In this paper we shall set down a comparable theory for buoyant parcels, but one which includes the whole history of the motion in a single formulation. The model to be discussed attempts to combine those features of the previous theories which seem physically the most real, and considers explicitly the changes in size produced by the competing processes of entrainment and the removal of fluid to the environment. Solutions will be presented for the instantaneous point source of buoyancy in turbulent surroundings having neutral and constant stable and unstable lapse rates, though the method could equally well be applied to finite sources and more general environments.

2. Description of the model

The present model is based on an extension of the picture of a thermal provided by laboratory experiments in still surroundings (e.g. Woodward 1959), and these observations will therefore be described in more detail. The mean motion inside and around a thermal is instantaneously very like that of a slightly flattened spherical vortex, and in fact this idea has been used explicitly by Levine (1959). The size is increasing with distance, however, so the streamlines are not closed, and external fluid enters the moving region. The fluid entering over the advancing front mixes vigorously with the buoyant fluid in the thermal, because of the unstable density gradient set up there, whereas that entering from the rear flows in more smoothly.

It is customary to define the size of a thermal in terms of the envelope containing all the original material of the thermal, but it is preferable for our purpose to use a definition based on the field of motion. At any instant there is a definite volume of fluid moving with the vortex, and we shall use the radius of the equivalent spherical vortex as a measure of the size at any height. At the turbulent front its boundary probably coincides with the edge of the buoyant material, but at the rear not all the advancing fluid is necessarily buoyant. The difference between these definitions is a minor one in still surroundings, but it becomes important when the environment is turbulent.

External turbulence will spread out the boundary between buoyant and non-buoyant fluid in such a way that it is no longer sharp, and some fluid originally in the thermal will escape into the environment. We shall consider this process in more detail below, but it is already clear that a definition of the edge based on the buoyant material would not be very satisfactory. It will still be true to say, however, that at any time there is a definite volume of fluid having a vortex-like circulation which is moving upwards with mean velocity $dz/dt = w$ say. The size of this vortex or its radius b may again be used to specify the size of the element, and it is the region within which the turbulence is governed by the effect of the buoyancy forces and the upward motion. We will retain the assumption that there is an inflow into this region at a rate proportional to the mean velocity w, and use the same numerical value for the entrainment constant α.

The removal of fluid from the element must now be specified in terms of a parameter which describes the environmental turbulence and does not depend on the interior motion. There seems to be no mechanism whereby internal turbulence can influence the diffusion beyond the boundary as it has been defined; some sort of 'erosion' by the environmental motion will be necessary to carry fluid away. We shall make a new assumption which is very similar to that on which the entrainment idea is based, namely, that the outflow velocity u_0 is proportional to some velocity specified by the turbulence in the environment, say the friction velocity u_* in a shear layer, or the root-mean-square turbulent velocity $(\overline{u^2})^{\frac{1}{2}}$. This is suggested by dimensional reasoning rather than any detailed picture of the flow, and its validity must be tested experimentally. It can perhaps be made more plausible by considering the following mechanistic description of the motion of particles of buoyant fluid which escape from the edge of the element. To be removed permanently a particle only needs to be moved sideways a short distance, to a streamline of the mean motion which does not enter the rear of the thermal, but which may lie only a small fraction of a radius from the boundary over a considerable part of its area. It is relevant therefore to consider the form of the diffusion law for short times, and in this range the turbulent particle velocities may be taken as nearly constant. If the 'edge' of the element could be regarded as a fixed source (which did not reabsorb fluid once it had been emitted) then the velocity of spread of particles from this source would be just the root-mean-square turbulent velocity; this should therefore be an upper limit to the efflux velocity from the edge.

In a constant stress layer in neutral surroundings u_* and $(\overline{u^2})^{\frac{1}{2}}$ are by definition constant, and they probably vary little with height over a considerable range of stability conditions (Panofsky & McCormick 1960). We shall assume that the velocity of outflow u_0 remains *exactly* constant with height always. The resulting equations of motion are similar in form to those used by Priestley, except that the dependence of mixing rates on element size is slightly different, and is included explicitly in the present formulation.

3. The derivation of the equations

Let the mean (potential) density within the element, regarded as a sphere of radius b, be $\rho(z)$, and that of the undisturbed environment be $\rho_0(z)$ at height z. Although the transition between buoyant and non-buoyant fluid will no longer be sharp, as discussed above, it seems reasonable to assume that buoyant fluid which escapes into the environment and gets left behind will quickly become so diluted that it will have a negligible effect on the properties of the environment. The stability conditions in the environment will therefore be unchanged with time and they may be specified by the parameter $G = -(g/\rho_1)\, d\rho_0/dz$ where ρ_1 is some reference density, say that at the height of the (virtual) source of the thermal and g is the acceleration due to gravity. The sign of G has been chosen to agree with an earlier definition; G is positive in stable, zero in uniform and negative in unstable surroundings.

Following Morton *et al.* (1956) the equation of continuity of mass may be written as

$$\frac{d}{dt}(\tfrac{4}{3}\pi b^3) = 4\pi b^2(\alpha w - u_0), \tag{1}$$

where b and w have already been defined, α is the entrainment constant and the second term on the right arises because of the new assumption that fluid is being removed from the element at a rate proportional to the surface area and to the characteristic velocity u_0. This is taken to be constant with height, even in conditions where the atmosphere is stably or unstably stratified. Equation (1) may immediately be integrated to give

$$b - b_0 = \alpha z - u_0 t, \tag{2}$$

where $b = b_0$ when $z = 0$ and $t = 0$. This reduces to the condition for a linear spread $b = \alpha z$ in the case where $b_0 = 0$ and $u_0 = 0$. Note that the existence of an outflow velocity always implies that the size of the element is less than predicted using the entrainment assumption alone.

With the same assumption about the inflow and outflow velocities, the momentum equation may be written

$$\frac{d}{dt}\left(\tfrac{4}{3}\pi b^3 \rho w\right) = \frac{4}{3}\frac{\pi}{C} b^3 g(\rho_0 - \rho) - 4\pi b^2 u_0 w\rho. \tag{3a}$$

This form implies that momentum equal to the mass flux times the mean velocity is being carried across the edge of the element by the fluid which leaves it, but that the fluid which enters from the surroundings has no mean upward momentum. The factor C is a virtual mass coefficient, introduced to take into account the acceleration of the fluid outside the 'edge' of the element which is also being set into motion by the buoyancy forces. For definiteness we shall take $C = \frac{3}{2}$, which is the value appropriate to a spherical vortex of constant size, and the equation ($3a$) becomes

$$\frac{d}{dt}(b^3 w) = \tfrac{2}{3} b^3 \Delta - 3b^2 u_0 w, \tag{3}$$

where $\Delta = g(\rho_0 - \rho)/\rho_1$, and density variations are taken to be small. Equation (3) thus takes some account of the motion in the environment near the element, although the main features of the behaviour should not be very sensitive to the exact form adopted.

Similarly the equation of continuity of density deficiency may be put into the form

$$\frac{d}{dt}\left(\tfrac{4}{3}\pi b^3(\rho_1 - \rho)\right) = 4\pi b^2 \alpha w(\rho_1 - \rho_0) - 4\pi b^2 u_0(\rho_1 - \rho), \tag{4a}$$

which can be simplified with the use of (1) to give

$$\frac{d(b^3\Delta)}{dt} = -b^3 w G - 3b^2 u_0 \Delta. \tag{4}$$

It will be convenient to use the equations (1), (3) and (4) with the dependent variables b, $M = b^3 w$ and $F = b^3\Delta$ to determine b, w and Δ as functions of time, and then to find the height z from (2). M is proportional to the momentum and F is related to the total buoyancy F_* by $F_* = \frac{4}{3}\pi F$. Three boundary conditions are required to define the solution; following Morton *et al.* we shall consider the case where b and M are zero at $t = 0$ and F is finite $= F_0$. Buoyant elements started off in other ways can often be thought of as arising from a virtual source some distance below the real source.

Using the three governing parameters provided by the problem, F_0 which is proportional to the buoyancy released initially, G describing the stability of the environment, and u_0 the characteristic velocity specifying the turbulence in the surroundings, the following transformations have been chosen to reduce the equations to a simple non-dimensional form:

$$\left.\begin{aligned} b &= (\tfrac{2}{3}\alpha)^{\frac{1}{2}} F_0^{\frac{1}{2}} u_0^{-1} b_1, \\ M &= \tfrac{2}{3}(\tfrac{2}{3}\alpha)^{\frac{1}{2}} F_0^{\frac{3}{2}} u_0^{-2} m, \\ F &= F_0 f, \\ t &= (\tfrac{2}{3}\alpha)^{\frac{1}{2}} F_0^{\frac{1}{2}} u_0^{-2} \tau, \\ z &= \left(\frac{2}{3\alpha}\right)^{\frac{1}{2}} F_0^{\frac{1}{2}} u_0^{-1} z_1, \\ G &= -\tfrac{9}{4}\alpha^{-1} F_0^{-1} u_0^4 \gamma, \end{aligned}\right\} \tag{5}$$

where b_1, m, f, τ, z_1 and γ are a complete set of non-dimensional variables.

The equations (1) to (4) become

$$\left.\begin{aligned}\frac{db_1}{d\tau} &= \frac{m}{b_1^3}-1,\\ b_1 &= z_1-\tau,\\ \frac{dm}{d\tau} &= f-\frac{3m}{b_1},\\ \frac{df}{d\tau} &= \gamma m-\frac{3f}{b_1},\end{aligned}\right\} \tag{6}$$

and the boundary conditions are $b_1 = 0$, $z_1 = 0$, $m = 0$ and $f = 1$ at $\tau = 0$. The non-dimensional parameter $\gamma = -\frac{4}{9}\alpha G F_0/u_0^4$ is a measure of the relative importance of buoyancy, density stratification and turbulence for the motion. An idea of the magnitude of the entrainment constant may be obtained from experimental results for the rate of spread in still surroundings, where it is about $\frac{1}{4}$, and this value will be retained here.

Numerical solutions of the set of equations (6) will be presented in the following sections. These were begun from the initial conditions $b_1 = 0, m = 0, f = 1, z_1 = 0$ at $\tau = 0$ using a starting series, and continued using a standard Runge–Kutta routine on the University of Sydney computing machine SILLIAC. Solutions have been obtained for a suitable range of the parameter γ, positive and negative, to test the effect of stability on the character of the motion. A wider range of solutions, for example starting with a finite radius and small velocity, can readily be obtained with little extra effort, but those which are presented here seem the most appropriate to illustrate the physical ideas which are being introduced in this paper.

4. The motion in neutral surroundings

First, the solutions with $\gamma = 0$ will be presented in detail. This corresponds to the case of an adiabatic atmosphere, in which the buoyancy released initially and the turbulent velocity in the surroundings are the only factors on which the motion can depend.

In figure 1 are shown b_1, z_1, Δ_1 and w_1 (where Δ_1 and w_1 are the non-dimensional equivalents of Δ and w) plotted against time, and in figure 2, b_1, Δ_1 and w_1 plotted against the non-dimensional height z_1. At first the motion within the element is dominant, and it spreads out, while Δ_1 decreases because of the dilution and w_1 decreases because the increasing momentum is distributed in a larger mass of fluid. Soon the effect of the removal of buoyant fluid becomes apparent: the rate of increase of radius decreases and at a non-dimensional height of $z_1 = 0{\cdot}350$ the radius achieves its maximum value of $b_1 = 0{\cdot}218$. At this height the inflow and outflow velocities are equal and $w_1 = 1$. Thereafter the solution suggests that the turbulence inside the element is less important than that in the environment: the element is eroded away, until at $z_1 = 0{\cdot}51$ approximately the radius becomes zero and the element has disappeared. This last part of the solution is the most difficult to defend physically, and an alternative procedure will be discussed in the next section. Reverting to the dimensional solution (5) and inserting the above numerical values and $\alpha = \frac{1}{4}$ we have therefore

$$\left.\begin{aligned} b_{\max} &= 0{\cdot}089 F_0^{\frac{1}{2}} u_0^{-1} \\ \text{at} \qquad z &= 0{\cdot}57 F_0^{\frac{1}{2}} u_0^{-1} \\ \text{and} \qquad z_{\max} &= 0{\cdot}83 F_0^{\frac{1}{2}} u_0^{-1}. \end{aligned}\right\} \tag{7}$$

These cannot be used for numerical prediction until a more definite meaning can be assigned to u_0, but they do show how the size of buoyant elements can depend on the buoyancy and the environmental turbulence.

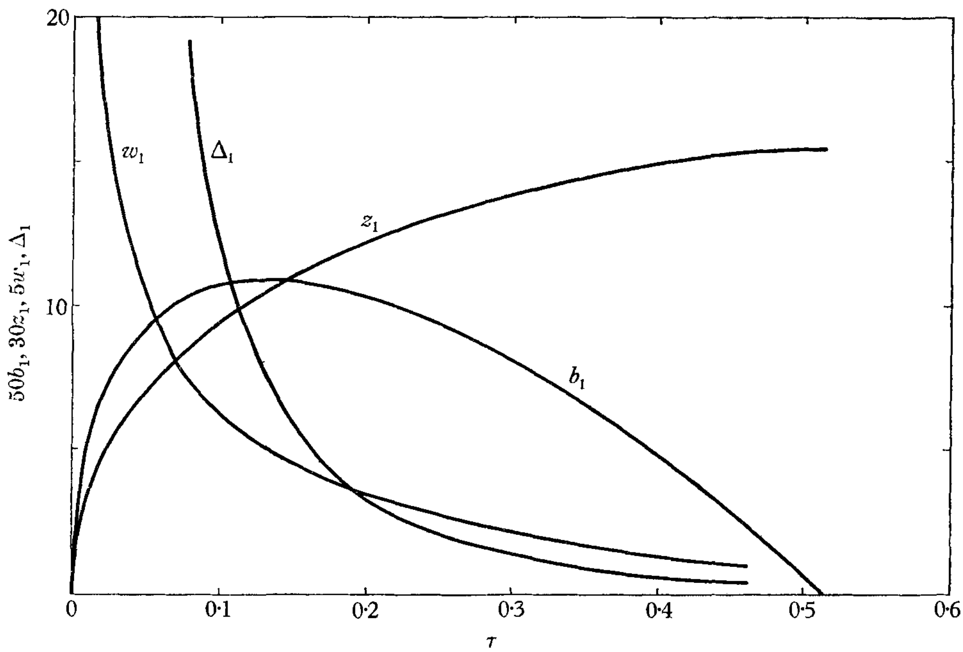

FIGURE 1. Non-dimensional radius b_1, height z_1, velocity w_1 and density difference Δ_1, plotted against time τ for a buoyant element starting with finite buoyancy and zero initial radius, rising through a turbulent environment.

A definite prediction arising out of this model, which is independent of the exact meaning of u_0, is that the horizontal scale of the buoyant elements should increase to a maximum at a certain height and then begin to decrease again. At the maximum, the ratio of diameter to height above the virtual origin will be

$$\frac{2b}{z} = \frac{2\alpha b_1}{z_1} = 0{\cdot}31, \tag{8}$$

using our numerical values.

5. The use of Priestley's model

The feature of the above solution which is perhaps most open to criticism is its continuation into the region where the radius is decreasing. At lower heights the model does seem to allow realistically for the growth of elements, when the turbulence within them is more important than that in the environment, but the opposite case, with environmental turbulence dominating, is more questionable. As an alternative, we might suppose that when the inflow and outflow velocities have become equal, the turbulence inside the element has the same intensity as

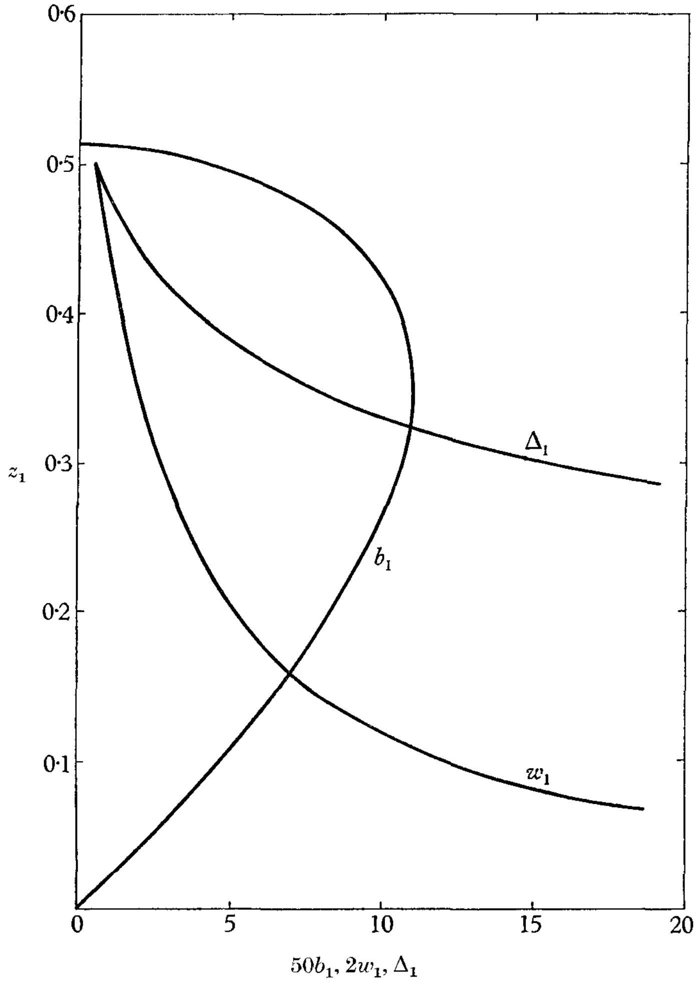

FIGURE 2. Non-dimensional radius, velocity and density difference plotted against height for the same conditions as in figure 1.

that in the environment, and can thereafter be regarded simply as part of the environmental turbulence.

This is the case discussed by Priestley (1953), and we can therefore use his results for the open parcel of constant size, taking as our initial conditions the radius, velocity and density difference at the height where the radius has its maximum. We have formulated the equations slightly differently, but they can easily be put into the form used by Priestley, for the general case of a linear lapse rate of any magnitude.

When $b_1 = b_m = \text{const.}$, $db_1/d\tau = 0$ and the last two equations of (6) may be written as

$$\left.\begin{aligned} \frac{dw_1}{d\tau} &= \Delta_1 - \frac{3w_1}{b_m}, \\ \frac{d\Delta_1}{d\tau} &= -3\frac{\Delta_1}{b_m} + \gamma w_1, \end{aligned}\right\} \tag{9}$$

where w_1 and Δ_1 have been defined as the non-dimensional representations of velocity and density difference. These are now in the form given by Priestley, with his 'mixing rates' k_1 for momentum and k_2 for heat replaced by

$$k_1 = k_2 = 3/b_m;$$

in his formulation these were proportional to $b_m^{-\frac{2}{3}}$. Priestley's solution shows that in our case elements are absolutely unstable if $\gamma \geqslant 9/b_m^2$ but that for values of γ less than this, the elements will come to rest at a height given by

$$z_p = \frac{|\Delta_m + (3w_m/b_m)|}{(9/b_m^2) - \gamma}; \tag{10}$$

the subscript m refers to conditions at the height at which b_1 achieves its maximum value b_m in our previous solution. As we saw earlier, $w_m = 1$.

More will be said about this after the results of the solution of (6) have been presented for stable and unstable environments. For the present, we should remark that in neutral surroundings an element continuing to rise in the manner suggested by Priestley will travel a further distance $z_p = 0{\cdot}115$ with its radius constant, to a final height of $z_1 = 0{\cdot}46$. This should be compared with the height $z_1 = 0{\cdot}51$ predicted in our earlier calculation where the size was allowed to decrease.

6. The solutions in stratified surroundings

Solutions of (6) have been obtained for a range of values of γ between ± 200, and the radii are plotted as a function of height in figure 3. Over this range of γ, all solutions behave like that with $\gamma = 0$ up to $z_1 = 0{\cdot}25$. With increasingly negative γ, corresponding to more stable conditions, the elements achieve a smaller maximum radius, and come to rest at lower heights. The practical importance of the stable case is of course questionable, since only rarely could a steady turbulent state be maintained under these conditions; the environment is more likely to be still, and this corresponds to the limit of infinite γ with zero u_0 for which the height has been shown (Morton *et al.* 1956) to be proportional to $F_0^{\frac{1}{4}} G^{-\frac{1}{4}}$. When γ is small and positive, or the environment is just unstable, the maximum size and the final height increase. At a value of γ of about 145, the behaviour changes markedly. When γ is larger than this elements increase in radius without limit, and they do not come to rest at a finite height. This condition for absolute instability is equivalent to that arising from Priestley's solution, since again it may be expressed as $\gamma b_m^2 \geqslant 9$. (This might be expected since the borderline case with b_m constant corresponds to the same pair of equations whichever formulation is used.)

This transition between markedly different behaviours occurs over only a small range of values of γ. This parameter is, by its definition $\gamma = -\frac{4}{9}\alpha G F_0/u_0^4$, a measure of the relative importance of buoyancy, the density gradient in the environment and the mixing rates; it is especially sensitive to the turbulent velocity in the surroundings. The results shown in figure 3 suggest that the whole

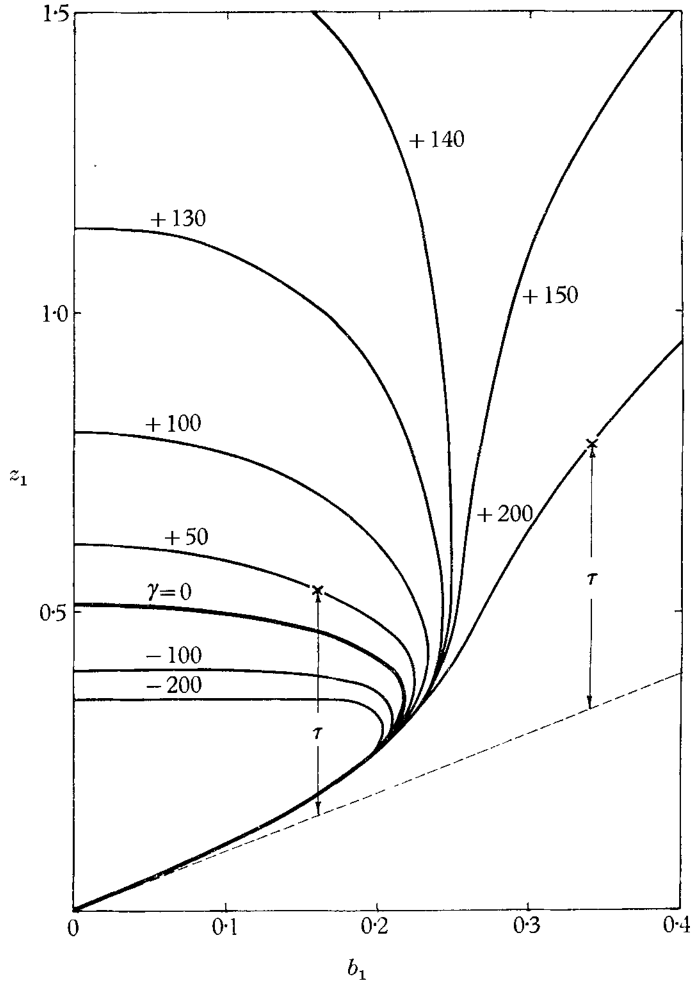

FIGURE 3. Showing dependence of the growth of buoyant elements in turbulent surroundings on the stability parameter $\gamma = -\frac{4}{9}\alpha G F_0/u_0^4$; positive values of γ correspond to unstable environments. The non-dimensional time τ at which given z_1 and b_1 are attained may also be obtained from this diagram using $\tau = z_1 - b_1$ (equation (2)), and this is illustrated for two points on the curves.

character of the motion of buoyant parcels could be dominated by small changes in environmental turbulence. For example the ability of individual buoyant parcels to reach the condensation level could be governed by the wind speed, through its influence on the magnitude of the turbulent velocities.

The modification suggested by Priestley's solution may again be applied to the results, using (10). In all stable cases, and in unstable environments up to the critical case of $\gamma = 145$, the final height may be obtained by this method, and the results are compared in figure 4 with the corresponding heights shown in figure 3. Between $\gamma = -200$ and $\gamma = +110$ the 'eroding' parcel achieves the greater height, and from $\gamma = 110$ to 145 the parcel which continues with constant

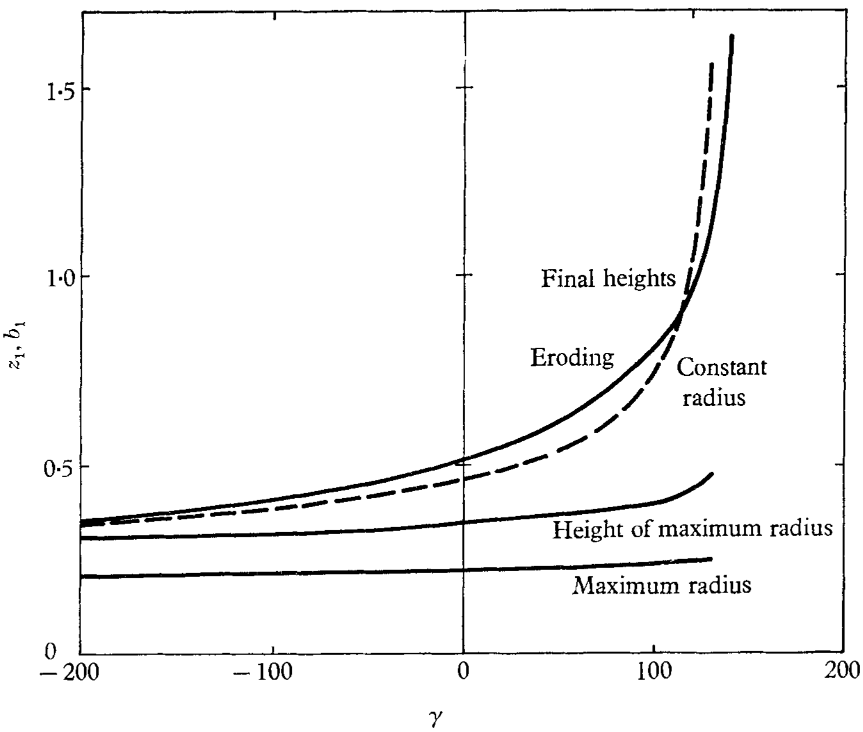

FIGURE 4. Comparison of the final heights attained by buoyant elements using the 'eroding' model and Priestley's results. Also shown are the maximum radius and the height at which this is attained as a function of the stability parameter.

radius does; the differences are not large. One other difference is that Priestley's solution predicts an oscillation of the elements about their final height in stable surroundings, with an asymptotic approach to the final height, whereas our solution shows a single approach to this height at a finite time.

Also shown on figure 4 are the maximum radius as a function of γ, and the height at which this is achieved. Both of these vary only slowly with γ, and the prediction made in equation (8) will be little changed by stability, provided of course that the limit $\gamma \approx 145$ is not exceeded.

7. An experimental test of the theory

The theoretical results have all depended on our basic assumption of a constant outflow velocity u_0, and it is desirable to check that this assumption is not widely at variance with experimental fact. A laboratory experiment has been devised which allows one prediction of the above theory to be tested directly, namely, the form of the dependence of the 'final height' on buoyancy and turbulent velocity in uniform surroundings (equation (7)). This experiment also allows an approximate comparison to be made between the deduced outflow velocity u_0 and the properties of the turbulence in the experimental tank.

The production of a deep constant-stress layer in the laboratory is difficult, so we have chosen another experimental situation to which the theory should be directly applicable. Experiments have been carried out in a large tank of water in which approximately isotropic turbulence was produced by agitation. At first the necessary stirring was effected by dragging a grid of cylindrical bars through the tank. This method would have the advantage that the properties of such grid turbulence—even the absolute intensity—are well known from wind-tunnel experiments, but it was found difficult to carry out in a stationary tank without introducing at the same time a large scale circulation. Instead, a regular array of sixteen jets was placed on the bottom of the 3 ft. × 3 ft. tank, and streams of large air bubbles blown from them up through the tank. Between the streams of air bubbles and the water dragged up by them countercurrents were formed, and when the air supply was turned off these jets broke down to give nearly homogeneous and isotropic turbulence.

This turbulence was of course decaying, and by carrying out convection experiments at various times during the decay it was possible to cover a wide range of turbulent intensities. The buoyant material used was coloured salt solution, with various densities so that we covered in all a range of total buoyancies of about a factor of six. This was released from the top of the tank with zero momentum by overturning a hemispherical cup 6 cm in diameter. Between successive experiments the tank was allowed to settle until any remaining motion was small. Air was bubbled for a fixed time at a standard rate, and the (arbitrary) zero of time taken as the instant when the air was turned off.

The behaviour of the buoyant element in a turbulent tank can be described as follows. Immediately it is released the characteristic vortex-like circulation is set up, and the element begins to spread out as it mixes with the surroundings; in addition it leaves a trail of marked fluid behind it. What we see visually as the 'edge' of the marked fluid depends, however, on the intensity of dye, which merges gradually into the surroundings. This is not the same as the *dynamical* edge which we have defined in this paper, and for this reason it is not easy to test the behaviour of the radius as a function of height. It is much simpler to follow the position of the vortex-like circulation as it moves through the tank, since our impression of this depends partly on the dye and partly on the motion. The final height z_{max} is taken to be the height at which this ordered motion is lost in the random motions; at this time the size of the ordered region is decreasing. In spite of the subjective element in this measurement we believe that the height can be recorded to the nearest 5 cm. The time of attaining this height is also noted; this is required in order to define the state of decay of turbulence in the tank, and not for comparison with the theoretical time of rise, so the slow approach of the elements to their final height does not lead to serious errors.

The state of the turbulence in the tank can easily be related to the time of decay as follows. In the initial period, which will certainly be the relevant one here, the turbulent energy decays according to the law

$$(\overline{u^2})^{-1} = c^2(t-t_0) \tag{11}$$

where $\overline{u^2}$ is, say, the mean-square turbulent velocity in one direction and t_0 is the virtual time at which the energy is infinite. The value of t_0, like the constant of

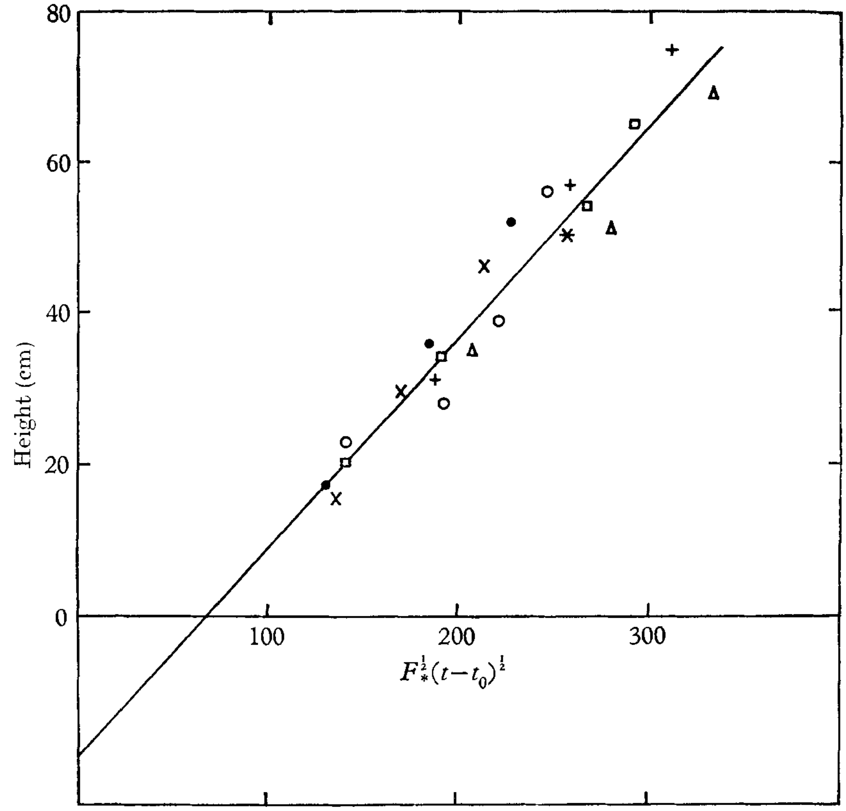

FIGURE 5. Showing experimental values for the 'final heights' attained by buoyant elements in turbulent surroundings, plotted against the function $F_*^{\frac{1}{2}}(t-t_0)^{\frac{1}{2}}$ of the total buoyancy F_* and the decay time $(t-t_0)$ which is suggested theoretically. The symbols refer to the following values of F_* ($cm^4\,sec^{-2}$): $\times$ 970, ● 1380, ○ 1940, □ 2560, △ 2910, + 3920, $*$ 5900.

proportionality c^2, must be determined experimentally. Using this expression, and assuming that u_0 is proportional to $(\overline{u^2})^{\frac{1}{2}}$, the relation we wish to test becomes

$$z_{\max} \propto F_*^{\frac{1}{2}}(t-t_0)^{\frac{1}{2}}. \tag{12}$$

Note again that t specifies the state of the turbulence, not the time during which convection is taking place. If an experiment is begun at time t_1 sec after the arbitrary origin of time and ends at t_2, then we take $t = \frac{1}{2}(t_1+t_2)$. During any one experiment the effects of decay were probably not serious, since t_2-t_1 was usually about $\frac{1}{3}t$ or less.

The results of over a hundred individual experiments are plotted in figure 5. Each point represents the average of five runs under identical conditions, and the different symbols refer to different total buoyancies. In order to plot the results in this form a value has to be assigned to t_0, and the one used, $t_0 = 5$ sec, has been chosen to give the best straight line fit to the experimental points. This process at the same time, and without further adjustment, gives a reasonable value for the position of the virtual source, at about three times the cup diameter above the water surface. Small changes in t_0 change the position of the virtual source slightly without altering the slope of the fitted straight line significantly.

The fact that the experimental points are all grouped closely about a straight line when plotted in this way shows that our basic hypothesis, that the outflow velocity is constant and proportional to the r.m.s. turbulent velocity in the environment, has led to a prediction which is consistent with experiment, at least over the limited range of the variables F_* and u_0 available to us. Our experiments do not of course provide a critical test of the suggested mechanism of mixing, but if other models were to lead to a different prediction, these could be tested against the present formulation by replotting the data of figure 5 in an appropriate form. If we use instead of F_* the variable F_0 defined in § 3 by $F_* = \frac{4}{3}\pi F_0$, then the measured slope of the line fitted in figure 5 gives the relation

$$z_{\max} = 0{\cdot}57 F_0^{\frac{1}{4}}(t-t_0)^{\frac{1}{2}}. \tag{13}$$

The final step, a quantitative comparison between (7) and (13), can only be carried out provided the constant c in (11) can be assigned, i.e. provided the absolute value of the turbulent velocity under the conditions of the experiment can be measured. This has been done in a subsidiary experiment carried out in the tank under identical conditions of turbulence, but without the buoyant element. Small plastic markers slightly denser than water were allowed to fall slowly through the turbulent tank, and photographed at intervals of 1 sec. From their successive positions in the interval $t = 15$–25 sec a value of $\overline{u^2}$ for the sideways motion was computed; it was found to be within 10% of $1{\cdot}0\,\mathrm{cm^2\,sec^{-2}}$ at the mean $t = 20$ sec.

Substituting $(\overline{u^2})^{\frac{1}{2}} = 1{\cdot}0$ and $(t-t_0) = 15$ in (11) gives $c = 0{\cdot}26$. Thus (13) may be rewritten in the form

$$z_{\max} = 2{\cdot}2 F_0^{\frac{1}{2}}(\overline{u^2})^{-\frac{1}{2}}. \tag{13a}$$

Finally, if we identify the final height measured with that predicted theoretically and called $z_{\max}$ in (7), we obtain the relation between the two velocities

$$u_0 \approx 0{\cdot}4(\overline{u^2})^{\frac{1}{2}}. \tag{14}$$

8. Discussion

The experiments reported above seem to give good support to our theoretical results in a uniform environment. The form of the dependence of the final height on the buoyancy and the state of the turbulence is well represented, and the numerical value of the deduced outflow velocity is reasonable, of the same order of magnitude but rather less than one component of the root-mean-square turbulent velocity in the surroundings. Since the final height is a quantity which depends on the whole history of the element, and there are no arbitrarily chosen parameters (apart from α, on which the predicted height depends only weakly), this good agreement may be regarded as an encouraging overall test of the theory. We can now proceed more confidently to examine the consequences of the same assumptions in stratified surroundings, although here unfortunately an experimental test is impracticable.

The most important result revealed by this analysis seems to be the extreme sensitivity of the buoyant motion to the level of environmental turbulence, as exhibited in figure 3, and this should remain at least qualitatively true even if u_0 is not strictly constant with height. Not only quantitative, but qualitative changes in the motion take place over a narrow range of values of the stability parameter $\gamma = -\frac{4}{9}\alpha G F_0/u_0^4$ and therefore over an even narrower range of the velocity u_0. For values of γ less than 145 the motion comes to rest at a finite height, whereas for values greater than this elements are absolutely unstable.

We can obtain from this limiting value of γ an estimate of the critical initial buoyancy which will lead to absolute instability in atmospheres having unstable lapse rates (compare with Priestley's 1953 estimate of a critical element *size*). Let us suppose for example that $(\overline{u^2})^{\frac{1}{2}} = 25\,\mathrm{cm\,sec^{-1}}$ so that u_0 is about $10\,\mathrm{cm\,sec^{-1}}$. For the range of potential temperature gradients covered by Priestley (i.e.

Potential temperature gradient °C cm^{-1}	10^{-7}	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}
F_* (cm^4 sec^{-2})	10^{14}	10^{13}	10^{12}	10^{11}	10^{10}	10^{9}
Computed radius r (metres)	220	100	46	22	10	$4\frac{1}{2}$
Priestley's R (metres)	1500	250	40	8	$1\frac{1}{2}$	$\frac{1}{4}$

TABLE 1

10^{-7} °C/cm, which exceeds the dry adiabatic rate by only a small fraction of its value, to 10^{-2} °C/cm, quite a strong lapse rate) we can compute approximate minimum values of F_* necessary for instability, and these are shown in table 1. Also shown are computed values of the radius of the elements of critical size, assuming they are spherical and have a temperature difference from their surroundings of 1 °C. Priestley's results are also shown for comparison, though the two sets are not strictly comparable.

The sizes suggested by this example are of the same order as those given by Priestley, though the range is smaller. The present formulation seems to have the advantage that the effect of a change in turbulent velocity in the surroundings is exhibited more directly; everything else being equal, the critical radius will be proportional to $u_0^{\frac{4}{3}}$.

Finally, let us estimate from the laboratory results the height to which parcels of warm air should rise in uniform surroundings in a typical case under the opposing influences of buoyancy and environmental turbulence. Taking $u_0 = 10\,\mathrm{cm\,sec^{-1}}$ as before and using (7) we find that for $F_* \approx 10^{12}$, z_{max} is about 400 metres, which is again a physically realistic result.

I am grateful to Mr F. K. Ball, Dr C. H. B. Priestley and Mr J. Warner for their helpful comments on an earlier draft of this paper.

REFERENCES

LEVINE, J. 1959 *J. Met.* **16**, 653.

MORTON, B. R., TAYLOR, G. I. & TURNER, J. S. 1956 *Proc. Roy. Soc.* A, **234**, 1.

PANOFSKY, H. A. & MCCORMICK, R. A. 1960 *Quart. J. Roy. Met. Soc.* **86**, 495.

PRIESTLEY, C. H. B. 1953 *Aust. J. Phys.* **6**, 279.

PRIESTLEY, C. H. B. 1956 *Quart. J. Roy. Met. Soc.* **81**, 139.

PRIESTLEY, C. H. B. 1960 *Turbulent Transfer in the Lower Atmosphere.* University of Chicago Press.

SCORER, R. S. 1957 *J. Fluid Mech.* **2**, 583.

WOODWARD, B. 1959 *Quart. J. Roy. Met. Soc.* **85**, 144.